An axiomatic characterization of the Theil inequality ordering

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Abstract

We characterize the Theil ordering of income inequality by means of ordinal axioms only.

1 Introduction

Economists have long been interested in the topic of income inequality. Typical issues include the evolution of income inequality over time in some particular region, the differences in income inequality across different regions, the effect of various policies on income inequality, and conversely, the effect of income inequality on various economic variables.[1] In order to

[1]See Goldberg and Pavcnik [18] for a recent survey on the effect of globalization on income inequality in developing countries. See also Helpman et al. [19] for a theoretical analysis of the effect of trade liberalization on income inequality.

address these and other similar questions one must first be able to measure income inequality, which is not a straightforward task.

The literature on the measurement of income inequality offers a plethora of inequality indices but the extent to which they are appropriate is not at all obvious. In order to compare the performance of different indices one may apply them to various income distributions and check whether or not these measures contradict one's intuitions about income inequality. For instance, they may be applied to two income distributions, one of which is believed to be more unequal than the other, and then discard all those indices that contradict our subjective judgment. Although this method may seem reasonable, it may not be very reliable. Discarding indices based on intuition is not the best scientific practice. Just as optical illusions may induce us to believe that one object is longer than another one while they are actually of equal length, so a false impression may induce us to believe that one income distribution is more unequal than another one, while in fact their level of inequality is the same.

Another, more cautious, way to evaluate inequality measures is to consider their properties at a more abstract level. We could make a list of properties that a reasonable inequality measure should satisfy and check which inequality measures do actually satisfy them. This method allows us to compare different indices in terms of the differential properties they do and do not satisfy, and has been successfully applied in the characterization of families of Gini-type indices, the Theil index, and the family of generalized entropy indices, among others. In particular, Bourguignon [7] and Foster [16] have shown that the Theil index of income inequality is the only index that satisfies several basic axioms as well as a simple decomposability property.

Some properties of inequality indices are uncontroversial, to the extent that they are considered as the defining properties of the bare concept of inequality measure. For example, take the Pigou-Dalton principle of transfers, which postulates that the transfer of income from a rich individual to a poorer one decreases inequality as long as the poor individual does not become richer than the rich one. Fields and Fey [15] consider this to be one of the basic axioms of inequality measurement. Other axioms, on the other hand, are less uncontroversial. For example, some of them require from an inequality index to be decomposable in some particular way. Specifically, given any partition of a society into two subsocieties, they require that the overall inequality be decomposable into the inequality between the subsocieties, and the inequality within them. Although very convenient, this decomposability is not at all an essential property of an inequality index.

It is important to bear in mind that some axioms are ordinal in nature, while others are cardinal. Ordinal axioms impose restrictions on how different income distributions are ranked. The Pigou-Dalton principle of transfers, for instance, is an ordinal property in that it compares two particular distributions and tells us which one is more unequal. It does not, however, relate to the magnitude of the inequality difference. Cardinal axioms, on the other hand, impose restrictions on the functional form of the index that is used to measure inequality. The decomposability property that both Bourguignon [7] and Foster [16] use to characterize the Theil index is cardinal, since it requires that the total inequality of a region be a weighted sum of the inequalities of its subregions and the inequality between these subregion. This property is lost if we apply a non-linear monotonic transformation to the index.

In this paper we characterize the Theil ordering of income inequality by means of ordinal axioms only. In particular, we strip the decomposability property used by Bourguignon [7] and Foster [16] from all its cardinal content, and retain only its ordinal content.

The rest of the paper is organized as follows. After giving a short review of the related literature in Section 2, we set up the model and list examples of prominent inequality indices in Section 3. Section 4 states the axioms and the main characterization theorem, the proof of which appears in Section 5. Section 6 concludes.

2 Related Literature

The axiomatic literature on inequality indices is quite vast. Weymark [29] defines a family of generalized Gini absolute inequality indices, and characterizes it in the class of societies with a fixed population. He also defines a family of generalized Gini relative inequality indices, which was axiomatically characterized by Ben Porath and Gilboa [5] within the class of societies with a fixed population and a fixed income. Yaari [30], Bossert [6] and Aaberge [1] provide characterizations of this family within a larger class of societies. Further characterizations of the Gini indices can be found in Thon [28], Donaldson and Weymark [13, 14], Yitzhaki [31], and Barret and Salles [4]. The family of generalized entropy measures has been studied by Cowell [10], Cowell and Kuga [11, 12], Shorrocks [23, 24], and Russell [22], to name a few. Finally, Atkinson [2] introduces and characterizes the family of Atkinson measures, which is further characterized by Lasso and Urrutia [21].[2]

One member of the family of general entropy indices is the Theil index, which has been introduced by Theil [26]. Theil [26, 27] shows that this index has the following useful property which, following Foster [16], we call Theil-Decomposability. Partition a society into two groups of income earners. We can define its *within-group* inequality as the weighted average of the income inequality levels of the two groups, the weights being the income shares of each group. We can also define the *between-group* inequality as the inequality level of the original society after smoothing the income of each group. In other words, between-group inequality is the inequality that would result if there was no within-group inequality. It turns out that no matter how we partition the original society, Theil's index measures its income inequality as the sum of the within-group and between-group inequalities.

Bourguignon [7] used this decomposability property to axiomatically characterize the Theil index. In particular, he shows that it is the only twice differentiable index that satisfies various uncontroversial axioms as well as Theil-Decomposability. Foster [16] shows that the requirement of twice differentiability can be replaced by continuity. In this paper

[2]For comprehensive surveys on income inequality measures, see Cowell [9] and Chakravarty [8].

we show, albeit on a larger class of societies, that we can replace the cardinal axiom of Theil-Decomposability by three weaker ordinal axioms and still obtain the Theil inequality ordering.

Our proof is very different from those of Bourguignon [7] and Foster [16]. Bourguignon heavily relies on the twice differentiability of the index. Foster, in turn, relies on Lee's [20] theorem to show that a particular simple function based on an index that satisfies Theil decomposability (and other uncontroversial axioms) must be a multiple of Shannon's measure of entropy.[3] In contrast, we rely on a well-known characterization of the logarithmic functions to show that a specific index that satisfies our axioms, restricted to particularly simple societies, is in fact a logarithmic function.[4] While Foster's proof consists mainly of showing that a Theil-Decomposable index must be a multiple of Theil's measure, the most burdensome part of our proof consists in showing that an inequality ordering that satisfies our axioms can be represented by a Theil-Decomposable index. Once this is done, showing that this index is in fact Theil's measure is less difficult.[5] Our result is reminiscent of Frankel and Volij's [17] characterization of the Mutual Information measure of segregation. The main difference is that Frankel and Volij, owing to an extensive use of an axiom of symmetry among different ethnic groups and one of invariance to splitting of groups, are able to prove their result without resorting to decomposability. In our case, however, since there are no multiplicity of groups, we have to add a decomposability axiom. Indeed, we will see that there are inequality indices that satisfy all our axioms except for decomposability.

[3]This simple function is the value of the index applied to a two-person society, one person earning a proportion t of the total income and the other one the remaining $1 - t$.

[4]These simple societies are ones where a proportion $1 - t$ of the population has no income at all and the remaining complementary proportion shares all of society's income evenly.

[5]We should point out that for this part of our proof we cannot rely on Foster's result because we deal with a class of societies that is larger than Foster's. Furthermore, his continuity axiom is different from ours.

3 Definitions

A *society* is a collection $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ where y_k, $k = 1, \ldots, K$ are nonnegative income levels, not all of them 0, and for each k, $n_k > 0$ is the mass of people with income level y_k. The elements (n_k, y_k) of a society are called *social classes.*

For a society $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$, we denote by $|S|$, the total level of income in S, and by $n(S)$ its total population. That is,

$$|S| = \sum_{k=1}^{K} n_k y_k \quad \text{and} \quad n(S) = \sum_{k=1}^{K} n_k.$$

Note that $|S| > 0$ and $n(S) > 0$.

For a society $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$, and $\alpha > 0$, αS denotes the society that is obtained from S by multiplying the number of people in each social class by α. That is, $\alpha S = \langle (\alpha n_1, y_1), \ldots, (\alpha n_K, y_K) \rangle$. We also denote by $\overline{S}$ the society $\left\langle \left(n(S), \frac{|S|}{n(S)} \right) \right\rangle$ that is obtained from S by redistributing S's income equally among its members. For any two societies S_1 and S_2, $S_1 \cup S_2$ denotes the concatenation of the two. We denote by $\mathcal{S}_n$, the set of all societies with population mass n, and by $\mathcal{S} = \cup_{n>0} \mathcal{S}_n$ the set of all societies. We also denote by $\mathcal{S}_+$ the subclass of societies where all individuals have strictly positive incomes.

An *inequality ordering* is a complete and transitive binary relation $\succcurlyeq$ on $\mathcal{S}$.[6] For any two societies $S_1$and S_2, $S_1 \succcurlyeq S_2$ means that S_1's income distribution is at least as unequal as S_2's. Some orderings can be represented by an *inequality index.* An inequality index is a function $I : \mathcal{S} \rightarrow \mathbb{R}$ that assigns to each society a real number, which represents the society's inequality level.

[6]We denote by $\succ$ and $\sim$ the asymmetric and symmetric parts of $\succcurlyeq$.

3.1 Examples of inequality indices

Example 1 The *Theil index*, $T : \mathcal{S} \to [0, \infty)$, is defined as follows:
for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$,

$$T(S) = \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \log_2 \left(n(S) \frac{y_k}{|S|} \right).$$

The *Theil ordering* is the ordering represented by the Theil index.

Note that the Theil index can be written as

$$T(S) = \log_2 n(S) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \log_2 \left(\frac{|S|}{y_k} \right)$$

or as the difference between the maximum entropy in a population of mass $n(S)$ and the entropy of the society.

Example 2 The *Second Theil index*, $T_0 : \mathcal{S}_+ \to [0, \infty)$, is defined as follows:
for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}_+$,

$$T_0(S) = \ln \left(\frac{|S|/n(S)}{\prod_{k=1}^{K} (y_k)^{\frac{n_k}{n(S)}}} \right).$$

The *Second Theil ordering* is the ordering represented by the Second Theil index.

Both T and T_0 belong to the family of generalized entropy indices, which are defined next.

Example 3 The *Generalized Entropy index*, $GE_\epsilon : \mathcal{S}_+ \to [0, \infty)$, is defined as follows:
For all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}_+$,

$$GE_\epsilon(S) = \begin{cases} \sum_{k=1}^{K} \frac{\left(\frac{n(S) y_k}{|S|}\right)^\epsilon - 1}{\epsilon^2 - \epsilon} \frac{n_k}{n(S)} & \text{if } \epsilon \neq 0, 1 \\ T_0(S) & \text{if } \epsilon = 0 \\ T(S) & \text{if } \epsilon = 1 \end{cases}$$

Example 4 The *Gini index*, $G : \mathcal{S} \to [0,1]$, is defined as follows:
Let $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$, and assume without loss of generality that the social classes are listed in increasing order of per capita income. That is, if $i < j$ then $y_i \leq y_j$. Then,

$$G(S) = 1 - \frac{\sum_{k=1}^{K} \frac{n_k}{n(S)} (\sum_{j=1}^{k-1} n_j y_j + \sum_{j=1}^{k} n_j y_j)}{|S|}$$

The *Gini ordering* is the ordering represented by the Gini index.

Example 5 The *Atkinson index*, $A_\epsilon : \mathcal{S}_+ \to [0,1]$, is defined as follows:
for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$,

$$A_\epsilon(S) = \begin{cases} 1 - \left(\sum_{k=1}^{K} \left(\frac{n(S) y_k}{|S|} \right)^{\epsilon} \frac{n_k}{n(S)} \right)^{1/\epsilon} & \text{if } \epsilon < 1, \epsilon \neq 0 \\ \\ 1 - \frac{n(S)}{|S|} \prod_{k=1}^{K} (y_k)^{\frac{n_k}{n(S)}} & \text{if } \epsilon = 0 \end{cases}$$

The *Atkinson orderings* are the ordering represented by the corresponding Atkinson indices.

Note that for all $S \in \mathcal{S}_+$, $T_0(S)$ is a monotone transformation of $A_0(S)$. Indeed, $T_0(S) = -\ln(1 - A_0(S))$. Therefore, the Second Theil ordering, T_0, and the Atkinson ordering, A_0, coincide.

Researchers are sometimes interested in decomposable inequality indices. Decomposable indices allow us to attribute total inequality to different factors. In particular, decomposable indices allow us to decompose total inequality into inequality *between* subsocieties and inequality *within* subsocieties. Bourguignon [7] and Foster [16] used the following version of decomposability in their characterizations of the Theil index.

Definition 1 [TD] We say that inequality index I is *Theil-decomposable* if for any two societies S_1 and S_2,

$$I(S_1 \cup S_2) = \frac{|S_1|}{|S_1 \cup S_2|} I(S_1) + \frac{|S_2|}{|S_1 \cup S_2|} I(S_2) + I(\overline{S_1} \cup \overline{S_2}). \tag{1}$$

The first two terms of (1) represent the inequality *within* S_1 and S_2. This inequality is the income-weighted average of the inequality of the two subsocieties as measured by I. The last term of (1) represents the inequality *between* S_1 and S_2. It is the inequality that would result if there was no inequality in either subsociety.

Note that Theil decomposability is a cardinal axiom. Nevertheless, it has very strong ordinal implications. In this paper we identify some of these ordinal implications and, together with other ordinal axioms, use them to characterize the Theil inequality ordering.

4 Axioms and the main result

We now present a set of axioms that an inequality ordering may satisfy. The first axiom embodies the idea that we are interested in *relative* measures of income inequality.

Definition 2 [HOM] We say that $\succcurlyeq$ satisfies *homogeneity* if for all $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ in $\mathcal{S}$, and for all $\alpha > 0$, we have $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \sim \langle (n_1, \alpha y_1), \ldots, (n_K, \alpha y_K) \rangle$.

Homogeneity states that only the relative distribution of income determines inequality. In other words, one does not need to know the units in which income is measured (dollars, euros, etc.) to determine whether one society has a more or less equal distribution than another one. It is easy to check that all the orderings listed in the previous section satisfy homogeneity.

The next axiom is similar to homogeneity. It states that it is not the absolute number of people who have any given income level that matters, but their proportion in the population.

Definition 3 [RI] We say that $\succcurlyeq$ satisfies *replication invariance* if for all $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ in $\mathcal{S}$, and for all $\alpha > 0$, we have $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \sim \langle (\alpha n_1, y_1), \ldots, (\alpha n_K, y_K) \rangle$.

Replication invariance, which is sometimes referred to as Dalton's principle of population, states that if we replicate a society by multiplying each individual by a fixed positive constant, then inequality remains unaffected. It can be easily checked that all the orderings listed in the previous sections satisfy replication invariance.

The previous two axioms dictate that a particular change in the society does not affect its income inequality. The next axiom, on the other hand, dictates that other changes do have a certain effect.

Definition 4 [TP] We say that $\succcurlyeq$ satisfies *the transfer principle* if for all egalitarian societies $\langle (n, y) \rangle$, and for all $\langle (n_1, y_1), (n_2, y_2) \rangle$ such that $n = n_1 + n_2$ and $ny = n_1 y_1 + n_2 y_2$ we have $\langle (n_1, y_1), (n_2, y_2) \rangle \succcurlyeq \langle (n, y) \rangle$, with equivalence ($\sim$) if and only if $y_1 = y_2$.

The transfer principle is essentially what Foster [16] calls directedness together with a nontriviality axiom. According to this principle, if one divides an egalitarian society into two social classes by transferring income from some individuals to others, then one obtains a new society with a more unequal distribution of income. It is easy to check that all the indices listed in the previous section represent orderings that satisfy the transfer principle.

The next axiom is an ordinal implication of the Theil-Decomposability axiom (1) used in Foster's [16] characterization of the Theil index.

Definition 5 [IND] We say that $\succcurlyeq$ satisfies *Independence*, if for all $S_1, S_2 \in \mathcal{S}$ such that $|S_1| = |S_2|$ and $n(S_1) = n(S_2)$, and for all societies $S \in \mathcal{S}$,

$$S_1 \succcurlyeq S_2 \Leftrightarrow S_1 \cup S \succcurlyeq S_2 \cup S.$$

Independence is essentially what is known as *subgroup consistency* which is closely related to the notion of *aggregativity* of an index (see Shorrocks [24, 25]). It says that if a given society is composed of two regions, and one of its regions' income becomes more unequally distributed, then the income distribution of the whole society becomes more unequal as well. The satisfaction of this axiom justifies the application of distributive policies in subregions in order to obtain results in the whole region. To illustrate, in order to reduce income inequality in Asia, one would want to apply a policy that reduces inequality in India. But this would be justified only if our measure of inequality satisfies IND. Otherwise, it may well be the case that by reducing inequality in India we end up increasing inequality in Asia. The Gini index does not satisfy Independence. To see this, consider the following societies, all of which have the same population and income: $S_1 = \langle(65,0),(95,384)\rangle$, $S_2 = \langle(60,0),(60,304),(40,456)\rangle$ and $S_3 = \langle(32,0),(128,285)\rangle$. It can be checked that while according to the Gini index S_2 is more unequal than S_1, adding S_3 to these societes reverses the order: $G(S_1 \cup S_3) > G(S_2 \cup S_3)$.

As mentioned above, IND is an ordinal implication of Theil-Decomposability. To see this, let $S_1, S_2 \in \mathcal{S}$ be two societies such that $|S_1| = |S_2|$ and $n(S_1) = n(S_2)$, and let $S \in \mathcal{S}$ be another society. Let $\alpha = \frac{|S_1|}{|S_1 \cup S|} = \frac{|S_2|}{|S_2 \cup S|}$, and $(1-\alpha) = \frac{|S|}{|S_1 \cup S|} = \frac{|S|}{|S_2 \cup S|}$. Then, if I is a Theil-Decomposable index,

$$I(S_1 \cup S) \geq I(S_2 \cup S) \Leftrightarrow \alpha I(S_1) + (1-\alpha) I(S) + I(\overline{S_1} \cup \overline{S}) \geq \alpha I(S_2) + (1-\alpha) I(S) + I(\overline{S_2} \cup \overline{S})$$

which, since $\overline{S_1} = \overline{S_2}$, is equivalent to $I(S_1) \geq I(S_2)$. Hence, I represents an ordering that satisfies IND.

The next axiom is another ordinal implication of Theil-decomposability.

Definition 6 [DEC] We say that $\succcurlyeq$ satisfies *Decomposability*, if for all four societies $S_1, S_2, S_3, S_4 \in \mathcal{S}$, such that

- $|S_1| = |S_2|$, and $n(S_1) = n(S_2)$ and
- $|S_3| = |S_4|$,

we have $(S_1 \cup S_3) \succcurlyeq (S_1 \cup S_4) \Leftrightarrow (S_2 \cup S_3) \succcurlyeq (S_2 \cup S_4)$.

Decomposability states the following. Suppose we want to compare two regions in terms of their income distribution. These two regions may not have the same population but they do have the same total income. Further suppose that these two regions share a common subregion. That is, their intersection is not empty. Think of Russia, which belongs both to Europe and to Asia, and assume that Europe and Asia have the same total income.

Decomposability dictates that whether or not one continent is more unequal than the other is independent of the income distribution in the common subregion. Continuing with our example, DEC states that whether or not Europe has a more unequal distribution than Asia is independent of how income is distributed within Russia.

Independence and decomposability are independent axioms. Indeed, the generalized entropy indices satisfy IND but not DEC (except, of course, for the Theil index, which satisfies both). Furthermore, the index $(-1)^{\lfloor |S| \rfloor} T(S)$ represents an ordering that satisfies DEC but not independence.

As mentioned above, DEC is weaker than Theil-Decomposability. To see this, let $S_1, S_2, S_3, S_4 \in \mathcal{S}$, such that $|S_1| = |S_2|$ and $n(S_1) = n(S_2)$, and $|S_3| = |S_4|$. Assume that the index I satisfies TD. Then, since $\overline{S_1} = \overline{S_2}$,

$$\begin{array}{rcll} (S_1 \cup S_3) & \succcurlyeq & (S_1 \cup S_4) & \Leftrightarrow \\ \frac{|S_1|}{|S_1 \cup S_3|} I(S_1) + \frac{|S_3|}{|S_1 \cup S_3|} I(S_3) + I(\overline{S_1} \cup \overline{S_3}) & \geq & \frac{|S_1|}{|S_1 \cup S_4|} I(S_1) + \frac{|S_4|}{|S_1 \cup S_4|} I(S_4) + I(\overline{S_1} \cup \overline{S_4}) & \Leftrightarrow \\ \frac{|S_2|}{|S_2 \cup S_3|} I(S_2) + \frac{|S_3|}{|S_2 \cup S_3|} I(S_3) + I(\overline{S_2} \cup \overline{S_3}) & \geq & \frac{|S_2|}{|S_2 \cup S_4|} I(S_2) + \frac{|S_4|}{|S_2 \cup S_4|} I(S_4) + I(\overline{S_2} \cup \overline{S_4}) & \Leftrightarrow \\ (S_2 \cup S_3) & \succcurlyeq & (S_2 \cup S_4) & \end{array}$$

which means that DEC is satisfied.

The last axiom is a technical but standard continuity requirement that states that "similar" societies have "similar" levels of income inequality.

Definition 7 [C] The inequality ordering $\succcurlyeq$ satisfies *continuity* if for all three societies S, S' and S'', the sets

$$\{\alpha \in [0,1] : \alpha S \cup (1-\alpha) S' \succcurlyeq S''\} \quad \text{and} \quad \{\alpha \in [0,1] : S'' \succcurlyeq \alpha S \cup (1-\alpha) S'\}$$

are closed.

We are now ready to state our main result.

Theorem 1 There is a unique inequality ordering defined on $\mathcal{S}$ that satisfies homogeneity, replication invariance, independence, decomposability, the transfer principle, and continuity. It is the Theil inequality ordering.

5 Proof of the main theorem

It is easy to check that the Theil ordering satisfies HOM, RI, TP, and C. It is well known that the Theil index satisfies Theil-Decomposability. Therefore it satisfies IND and DEC as well.

Before we show that no other inequality ordering satisfies the above axioms, it will be convenient to define an auxiliary property. As we will show shortly, it is an implication of the transfer principle and independence. It requires that a division of any social class into two different social classes derived from a transfer of income from one group of individuals to another results in a society with more unequal distribution of income.

Definition 8 [SCDP] We say that $\succcurlyeq$ satisfies the *social class division property* if whenever S' is obtained from S by means of a subdivision of a social class $(n_k, y_k) \in S$ into two social classes $(n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2})$ such that $n_k = n_{k_1} + n_{k_2}$ and $n_k y_k = n_{k_1} y_{k_1} + n_{k_2} y_{k_2}$, we have that $S' \succcurlyeq S$, with equivalence $(\sim)$ if and only if $y_{k_1} = y_{k_2}$.

Claim 1 If the inequality order $\succcurlyeq$ satisfies independence, and the transfer principle, then it also satisfies the social class division property.

Proof : Let $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ be any society and let $S' = (S \backslash \langle (n_k, y_k) \rangle) \cup \langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle$ be the society that is obtained from S by means of a subdivision of a social class $(n_k, y_k) \in S$ into two social classes $(n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2})$ such that $n_k = n_{k_1} + n_{k_2}$ and $n_k y_k = n_{k_1} y_{k_1} + n_{k_2} y_{k_2}$. By the transfer principle,

$$\langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle \succcurlyeq \langle (n_k, y_k) \rangle .$$

By independence,

$$(S \backslash \langle (n_k, y_k) \rangle) \cup \langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle \succcurlyeq (S \backslash \langle (n_k, y_k) \rangle) \cup \langle (n_k, y_k) \rangle$$

or, $S' \succcurlyeq S$. □

Now let $\succcurlyeq$ be an inequality ordering on $\mathcal{S}$ that satisfies homogeneity, replication invariance, independence, decomposability, the transfer principle, and continuity. We will show that is represented by the Theil index. Let $S_0 = \langle(1,1)\rangle$ be the society with population mass 1 and a uniformly distributed income of one, and let $S_{1/2} = \langle(1/2,0),(1/2,2\rangle$ be the society with population mass 1, in which half of the population has income 0, and the other half has income 2. Note that $S_{1/2}$ has population 1 and income 1.

Lemma 1 All societies where total income is uniformly distributed have the same degree of income inequality. Further, for all societies $S \in \mathcal{S}$, $S \succcurlyeq S_0$.

Proof : Let S be a society with uniformly distributed income. By HOM, RI, and SCDP, $S \sim S_0$. Also let $S = \langle(n_1, y_1), \ldots, (n_K, y_K)\rangle \in \mathcal{S}$ be an arbitrary society, and let S^k be the society that results from combining social classes 1 to k, into one social class $\left(\sum_{i=1}^k n_i, \sum_{i=1}^k n_i y_i / \sum_{i=1}^k n_i\right)$. By SCDP, $S^k \succcurlyeq S^{k+1}$. Therefore, $S = S^1 \succcurlyeq S^K$. But S^K has only one social class, and hence income is uniformly distributed there. $\square$

Lemma 2 Let S' be a society with population 1 and income 1 such that $S' \succ S_0$. Then, for $0 \leq \alpha < \beta < 1$,

$$\beta S' \cup (1-\beta)S_0 \succ \alpha S' \cup (1-\alpha)S_0$$

Proof : By RI, $(\beta-\alpha)S' \succ (\beta-\alpha)S_0$. By IND,

$$\alpha S' \cup (\beta-\alpha)S' \cup (1-\beta)S_0 \succ \alpha S' \cup (\beta-\alpha)S_0 \cup (1-\beta)S_0.$$

By SCDP,

$$\beta S' \cup (1-\beta)S_0 \succ \alpha S' \cup (1-\alpha)S_0.$$

$\square$

Lemma 3 Let S' be a society with population 1 and income 1 such that $S' \succ S_0$. Then, for any society $S \in \mathcal{S}$ such that $S' \succcurlyeq S \succcurlyeq S_0$, there is a unique $\alpha^* \in [0,1]$ such that

$$S \sim \alpha^* S' \cup (1-\alpha^*) S_0$$

Proof: By C, the sets $\{\alpha \in [0,1] : \alpha S' \cup (1-\alpha) S_0 \succcurlyeq S\}$ and $\{\alpha \in [0,1] : S \succcurlyeq \alpha S' \cup (1-\alpha) S_0\}$ are closed. Since $S' \succcurlyeq S \succcurlyeq S_0$, they are not empty. Since $\succcurlyeq$ is complete, their union is $[0,1]$. Therefore, since the unit interval is connected, the intersection of the two sets is not empty. By Lemma 2, this intersection must contain a single element. This single element is the α^* we are looking for. $\square$

Lemma 4 For any society $S \in \mathcal{S}$ such that $|S| = n(S) = 1$, there is a unique $\alpha^* \geq 0$ such that

$$S \cup \alpha^* S_0 \sim S_0 \cup \alpha^* S_{1/2}.$$

Proof : If $S_{1/2} \succcurlyeq S \succcurlyeq S_0$, then by Lemma 3 there is a unique $\alpha^* \in [0,1]$ such that $S \sim \alpha^* S_{1/2} \cup (1-\alpha^*) S_0$. Then, by IND and SCDP,

$$\begin{aligned} S \cup \alpha^* S_0 &\sim \alpha^* S_{1/2} \cup (1-\alpha^*) S_0 \cup \alpha^* S_0 \\ &\sim \alpha^* S_{1/2} \cup S_0. \end{aligned}$$

If, on the other hand, $S \succ S_{1/2}$, by Lemma 3 there is a unique $\beta^* \in [0,1]$ such that $\beta^* S \cup (1-\beta^*) S_0 \sim S_{1/2}$. Since $S_{1/2} \succ S_0$, $\beta^* > 0$. Then,

$$\begin{aligned} S \cup \tfrac{(1-\beta^*)}{\beta^*} S_0 &\sim \tfrac{1}{\beta^*} S_{1/2} && \text{by RI} \\ S \cup \tfrac{(1-\beta^*)}{\beta^*} S_0 \cup S_0 &\succcurlyeq \tfrac{1}{\beta^*} S_{1/2} \cup S_0 && \text{by IND} \\ S \cup \tfrac{1}{\beta^*} S_0 &\succcurlyeq \tfrac{1}{\beta^*} S_{1/2} \cup S_0 && \text{by SCDP.} \end{aligned}$$

Therefore, $\frac{1}{\beta^*}$ is the α^* we are looking for. $\square$

Lemma 4 allows us to define an index $r : \mathcal{S} \to \mathbb{R}$ by

$$r(S) = \alpha,$$

where α is the unique number that satisfies $\widehat{S} \cup \alpha S_0 \sim S_0 \cup \alpha S_{1/2}$, and $\widehat{S}$ is the society that is obtained from S by normalizing its population and income to 1.

Lemma 5 The index r represents the inequality order $\succcurlyeq$.

Proof : Let S and S' be two societies and assume $S' \succcurlyeq S$. By RI and HOM we can assume that $|S| = |S'| = 1$ and $n(S) = n(S') = 1$. Let α and α' be defined by

$$S \cup \alpha S_0 \sim S_0 \cup \alpha S_{1/2} \tag{2}$$

and

$$S' \cup \alpha' S_0 \sim S_0 \cup \alpha' S_{1/2}. \tag{3}$$

We need to show that $\alpha' \geq \alpha$. Assume by contradiction that $\alpha > \alpha'$. Then,

$$\begin{array}{rcll} S' \cup \alpha S_0 & \succcurlyeq & S \cup \alpha S_0 & \text{by IND} \\ & \sim & S_0 \cup \alpha S_{1/2} & \text{by (2)} \\ & \sim & S_0 \cup \alpha' S_{1/2} \cup (\alpha - \alpha') S_{1/2} & \text{by SCDP} \\ & \succ & S_0 \cup \alpha' S_{1/2} \cup (\alpha - \alpha') S_0 & \text{by IND and } S_{1/2} \succ S_0, \\ & \sim & S' \cup \alpha' S_0 \cup (\alpha - \alpha') S_0 & \text{by (3) and IND} \\ & \sim & S' \cup \alpha S_0 & \text{by SCDP,} \end{array}$$

which cannot be true. □

The next proposition implies that the index r satisfies Theil-Decomposability.

Proposition 1 Let S and S' be two societies. Then

$$r(S \cup S') = \frac{|S|}{|S \cup S'|} r(S) + r\left(\left\langle \left(n(S), \frac{|S|}{n(S)}\right) \right\rangle \cup S'\right)$$

Proof : Let S and S' be two societies, with populations n and m, respectively. By RI and HOM, we can assume without loss of generality that $n + m = 1$, and $|S \cup S'| = 1$. Let $\widehat{S}$ be the society that is obtained from S by normalizing its population and income so that they are both equal to 1. Let γ be defined by

$$\left(\left\langle n, \frac{|S|}{n} \right\rangle \cup S'\right) \cup \gamma S_0 \sim S_0 \cup \gamma S_{1/2}. \tag{4}$$

Similarly, let α be defined by

$$\widehat{S} \cup \alpha S_0 \sim S_0 \cup \alpha S_{1/2}. \tag{5}$$

We need to show that

$$(S \cup S') \cup (|S|\,\alpha + \gamma) S_0 \sim S_0 \cup (|S|\,\alpha + \gamma) S_{1/2}. \tag{6}$$

Pick $k \in \mathbb{N}$ such that $k > 1 + \alpha\,|S|$. Denote $S^*_{1/2} = n\left\langle (1/2, 0), (1/2, \frac{2|S|}{n}) \right\rangle$ and $S^*_0 = \left\langle (n, \frac{|S|}{n}) \right\rangle$. These two societies have population n and income $|S|$. It follows from (5), using RI and HOM, that

$$S \cup \alpha S^*_0 \sim S^*_0 \cup \alpha S^*_{1/2}. \tag{7}$$

Equation (4) can be written as

$$(S^*_0 \cup S') \cup \gamma S_0 \sim S_0 \cup \gamma S_{1/2}.$$

Therefore,

$$\begin{aligned}
S^*_0 \cup \overbrace{S' \cup \gamma S_0 \cup (k-1)(S_0 \cup \gamma S_{1/2})}^{Z_3} &\sim k(S_0 \cup \gamma S_{1/2}) && \text{by IND and SCDP} \\
&\sim \tfrac{k}{|S|}(S^*_0 \cup \gamma S^*_{1/2}) && \text{by HOM and RI} \\
&\sim S^*_0 \cup \overbrace{\frac{k}{|S|}((\underbrace{1 - \frac{|S|}{k})}_{>0} S^*_0 \cup \gamma S^*_{1/2})}^{Z_4} && \text{by SCDP.}
\end{aligned}$$

Note that since $|S| < 1$ and $\alpha \geq 0$, our choice of k implies that $|S|/k < 1$, and therefore subsociety Z_4 is well-defined. Since $|Z_3| = |Z_4| = |S'| + k\gamma + (k-1)$, by DEC,

$$S \cup \overbrace{S' \cup \gamma S_0 \cup (k-1)(S_0 \cup \gamma S_{1/2})}^{Z_3} \sim S \cup \overbrace{\frac{k}{|S|}((1 - \frac{|S|}{k}) S^*_0 \cup \gamma S^*_{1/2})}^{Z_4}.$$

By SCDP,

$$
\begin{aligned}
S \cup S' \cup \gamma S_0 \cup (k-1)(S_0 \cup \gamma S_{1/2}) &\sim S \cup \alpha S_0^* \cup \tfrac{k}{|S|}(\underbrace{(1 - \frac{(1+\alpha)|S|}{k})}_{>0} S_0^* \cup \gamma S_{1/2}^*) \\
&\sim S_0^* \cup \alpha S_{1/2}^* \cup \tfrac{k}{|S|}((1 - \tfrac{(1+\alpha)|S|}{k}) S_0^* \cup \gamma S_{1/2}^*) \\
&\sim |S| \left(S_0 \cup \alpha S_{1/2}\right) \cup (k - (1+\alpha)|S|) S_0 \cup k\gamma S_{1/2} \\
&\sim S_0 \cup (\alpha |S| + \gamma) S_{1/2} \cup (\underbrace{k - 1 - \alpha |S|}_{>0}) S_0 \cup (k-1)\gamma S_{1/2}
\end{aligned}
$$

where the second line follows from (7) and IND, the third line from HOM and RI, and the last one from SCDP. On the other hand, by SCDP

$$
S \cup S' \cup \gamma S_0 \cup (k-1)(S_0 \cup \gamma S_{1/2}) \sim S \cup S' \cup (\alpha |S| + \gamma) S_0 \cup (k - 1 - \alpha |S|) S_0 \cup (k-1)\gamma S_{1/2}.
$$

As a result, denoting $k^* = (k - 1 - \alpha |S|)$, we obtain

$$
S \cup S' \cup (\alpha |S| + \gamma) S_0 \cup \left(k^* S_0 \cup (k-1)\gamma S_{1/2}\right) \sim S_0 \cup (\alpha |S| + \gamma) S_{1/2} \cup \left(k^* S_0 \cup (k-1)\gamma S_{1/2}\right)
$$

Since $S \cup S' \cup (\alpha |S| + \gamma) S_0$ and $S_0 \cup (\alpha |S| + \gamma) S_{1/2}$ have the same population and income, we can apply IND and obtain

$$
S \cup S' \cup (\alpha |S| + \gamma) S_0 \sim S_0 \cup (\alpha |S| + \gamma) S_{1/2},
$$

which is what we wanted to prove. □

Corollary 1 Let $S_1, \ldots, S_K$ be K societies. Also let $S = \bigcup_{k=1}^{K} S_k$. Then

$$
r(S) = \sum_{k=1}^{K} \frac{|S_k|}{|S|} r(S_k) + r\left(\bigcup_{k=1}^{K} \left\langle \left(n(S_k), \frac{|S_k|}{n(S_k)}\right) \right\rangle \right).
$$

Proof : The proof is by induction and is left to the reader. □

We now define a class of simple societies. For each $\alpha \in (0,1)$, let $S_\alpha = \langle(\alpha, 0), (1-\alpha, 1/(1-\alpha)\rangle$ be the society with population mass 1, in which a proportion α

of the population has income 0, and the proportion $(1-\alpha)$ of the population has income $1/(1-\alpha)$. The next proposition shows that r, when applied to these societies, induces a well-known function.

Proposition 2 For all $\alpha \in (0,1]$, $r(S_{1-\alpha}) = -\log_2 \alpha$.

Proof : Let $h : (0,1] \to \mathbb{R}$ be defined by $h(\alpha) = r(S_{1-\alpha})$. By definition of r,

$$h(\alpha) \geq 0 \quad \text{for all } \alpha \in (0,1]. \tag{8}$$

Also,

$$h(1/2) = r(1/2) = 1. \tag{9}$$

We will now show that

$$h(pq) = h(p) + h(q) \quad \text{for all } p, q \in (0,1]. \tag{10}$$

To see this, note that

$$\begin{aligned} S_{1-pq} &= \left\langle (1-pq, 0), \left(pq, \tfrac{1}{pq}\right) \right\rangle \\ &\sim \left\langle (1-q, 0), (q(1-p), 0), \left(pq, \tfrac{1}{pq}\right) \right\rangle && \text{by SCDP} \\ &= \left\langle (q(1-p), 0), \left(pq, \tfrac{1}{pq}\right) \right\rangle \cup \langle (1-q, 0) \rangle . \end{aligned}$$

Therefore, by Proposition 1,

$$\begin{aligned} r(S_{1-pq}) &= r\left(\left\langle (q(1-p), 0), \left(pq, \tfrac{1}{pq}\right) \right\rangle\right) + r\left(\left\langle \left(q, \tfrac{1}{q}\right), (1-q, 0) \right\rangle\right) \\ &= r\left(\left\langle (1-p, 0), \left(p, \tfrac{1}{p}\right) \right\rangle\right) + r\left(\left\langle \left(q, \tfrac{1}{q}\right), (1-q, 0) \right\rangle\right) && \text{by HOM and RI} \\ &= r\left(S_{1-p}\right) + r(S_{1-q}), \end{aligned}$$

which shows that (10) holds. It is known that the only function on $(0,1]$ that satisfies (8-10) is $-\log_2$.[7] □

[7]See Theorem 0.2.5 in Aczél and Daróczy [3].

Proposition 3 The index r is the Theil index.

Proof : Let $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$ be a society. We need to show that $r(S) = T(S)$. If $K = 1$, the result is obvious. So assume $K \geq 2$. By RI we can assume without loss of generality that $n(S) = 1$. Similarly, by HOM we can assume without loss of generality that $\sum y_k = 1$. Therefore $|S|^2 < |S| = \sum n_k y_k < 1$. Also, $y_k|S| < 1$ for $k = 1, \ldots K$. Define

$$S^K = \bigcup_{k=1}^{K} \left\langle (n_k(1 - y_k|S|)), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle.$$

That is, S^K is the result of replacing social classes (n_k, y_k), $k = 1, \ldots, K$, in S by $\left\langle (n_k(1 - y_k|S|)), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle$, respectively. Therefore, by Corollary 1,

$$r(S^K) = \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(\left\langle (n_k(1 - y_k|S|)), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle\right) + r(S)$$

which can be written as

$$r(S) = r(S^K) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(\left\langle (n_k(1 - y_k|S|), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle\right).$$

Note that by RI, and HOM,

$$\begin{aligned}\left\langle (n_k(1 - y_k|S|), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle &\sim \left\langle (1 - y_k|S|, 0), \left(y_k|S|, \frac{1}{|S|}\right)\right\rangle \\ &\sim \left\langle (1 - y_k|S|), 0), \left(y_k|S|, \frac{1}{y_k|S|}\right)\right\rangle = S_{1-y_k|S|}.\end{aligned}$$

Also, by SCDP,

$$\begin{aligned}S^K &= \cup_{k=1}^{K} \left\langle (n_k(1 - y_k|S|)), 0), \left(n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle \\ &\sim \left\langle \left(\sum_{k=1}^{K} n_k(1 - y_k|S|), 0\right), \left(\sum_{k=1}^{K} n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle \\ &\sim \left\langle \left(\sum_{k=1}^{K} n_k - \sum_{k=1}^{K} n_k y_k|S|), 0\right), \left(\sum_{k=1}^{K} n_k y_k|S|, \frac{1}{|S|}\right)\right\rangle \\ &\sim \left\langle (1 - |S|^2), 0), \left(|S|^2, \frac{1}{|S|}\right)\right\rangle = S_{1-|S|^2}.\end{aligned}$$

Therefore

$$\begin{aligned} r(S) &= r(S_{1-|S|^2}) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(S_{1-y_k|S|}\right) \\ &= \sum_{k=1}^{K} \frac{n_k y_k}{|S|}\left(r(S_{1-|S|^2}) - r\left(S_{1-y_k|S|}\right)\right) \\ &= \sum_{k=1}^{K} \frac{n_k y_k}{|S|}\left(\log_2 \frac{1}{|S|^2} + \log_2 y_k|S|\right) \\ &= \sum_{k=1}^{K} \frac{n_k y_k}{|S|}\left(\log_2 \frac{y_k}{|S|}\right) \\ &= T(S). \end{aligned}$$

□

6 Conclusions

We have axiomatically characterized the Theil ordering of income inequality. In addition to the uncontroversial axioms of homogeneity, replication invariance, the transfer principle and a standard continuity property, we appealed to an independence and to a decomposability axioms. These two axioms are ordinal implications of Theil Decomposability, the central axiom in Bourguignon [7] and Foster [16] in their characterization of the Theil index. To the best of our knowledge, this is the first fully ordinal characterization of this index.

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